Component Mode Synthesis for Damped Structures

Kenji Kubomura*
Beloit Manhattan, Inc., Clarks Summit, Pennsylvania

This paper formulates and investigates component mode synthesis methods for structures with the nonproportional symmetric damping matrix. The use of three different types of modes is discussed: complex free-free, cantilever, and hybrid. For ease of computation, the reduction transformation equations are expressed in real numbers with partitions from the stiffness, damping and mass matrices, and the real and complex parts of the component modes of the desired frequency ranges. The methods can retain the content of any frequency range of interest in the reduced component matrices. Numerical results validate these reduction transformation equations.

Nomenclature

$d_{\ell m}$ = damping coefficient
$E_{ijk\ell}$ = elastic constant
f_i = components of the force vector $\boldsymbol{f}$
g_i = adjoint quantity of f_i
h = component eigenvector
ℓ = component eigenvalue
L = system eigenvalue
m = mass density
p_i = components of independent variable vector $\boldsymbol{p}$
q_i = adjoint quantity of p_i
Q = system eigenvector
u_i = components of the displacement vector $\boldsymbol{u}$
v_i = adjoint quantity of u_i
$\boldsymbol{Z}, \boldsymbol{X}$ = independent variable vector
Z_i, X_i = components of vector $\boldsymbol{Z}$ or $\boldsymbol{X}$
$[\ ]$ = matrix
$[D]$ = damping matrix
$[\boldsymbol{q}]$ = matrix whose columns are eigenvectors $\boldsymbol{q}$
$[I]$ = unit matrix
$[K]$ = stiffness matrix
$[M]$ = mass matrix
$[O]$ = null matrix
$[\ell]$ = diagonal matrix with its diagonal terms being eigenvalues ℓ
$[\ ]^{-1}$ = inverse matrix
$(\ )^T$ = transpose of a vector or matrix
$\bar{(\ )}$ = quantities pertinent to cantilever mode
$(\ )_b$ = quantities pertinent to interface degrees of freedom
$(\ )_i$ = quantities pertinent to internal (noninterface) degrees of freedom
$^{H}(\ )$ = association with component eigenvalues larger than system eigenvalue of interest (higher frequency)
$^{L}(\ )$ = association with component eigenvalues smaller than system eigenvalue of interest (lower frequency)
$^{C}(\ )$ = association with component eigenvalues comparable to system eigenvalue of interest
$^{R}(\ )$ = association with zero eigenvalues (rigid body mode)
$^{e}(\ )$ = association with nonzero eigenvalues (elastic mode)
$\dot{(\ )}$ = derivative with respect to time
$(\ )_{,i}$ = derivative with respect to spatial coordinate x_i

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*Assistant Director of Research and Development.

Introduction

A LARGE system structure consists of several components, each represented by its finite-element equation of motion. In many cases, the component equation size is very large and is reduced by modal synthesis techniques. The reduced component equations are then coupled to construct the overall system equations. This method substantially decreases the number of system equations. Aerospace and civil structures are usually lightly damped. Therefore, proportional damping is assumed for the calculation and survey of component natural modes. These natural modes are used in a reduction transformation of the component finite-element equations by modal synthesis. Reference 1 presents a procedure for developing methods to reduce the size of the component finite-element equations, as well as selecting which component modes should be retained or approximated according to the relative magnitude of the component eigenvalues vs the system eigenvalues of interest.

Recently, several investigations have been made into the component mode synthesis of linear structures with nonproportional damping. Hasselman and Kaplan[2] developed a coordinate transformation by using fixed interface substructure natural modes to reduce component degrees of freedom. It is not clear whether or not the substructure portion of the system eigenvectors of interest can be represented adequately by these modes. Chung and Craig[3] introduced complex residual attachment modes to account for the contribution from the truncated higher free-free substructure modes. These residual attachment modes require the calculation of all substructure free-free complex modes. To facilitate these calculations, Howsman and Craig[4] approximated the total flexibility by the inverse of the statically determinant stiffness matrix. Numerical results show the method to be accurate for the case of a skew symmetric damping matrix, but it lacks rigorous proof. Hale[5] discussed the reduction of overall system degrees of freedom through the iteration of several trail vectors. His method does not allow the reduction of each substructure matrix in the substructure level with the assessment of the contribution from the substructure to the system modes of interest.

This paper extends the previous work[1] by including structures with nonproportional damping and developing the associated coordinate transformation equations (reduction transformation equations). These transformation equations are used to reduce the size (degrees of freedom) of the component equations from which the system equations of motion are constructed by enforcing compatibility and equilibrium at the component boundaries. In this method, the reduction transformation equations do not require component modes of eigenvalues that are far larger or smaller than the system

eigenvalues of interest. It should be mentioned that these equations are used to reduce each component matrix equation independently from matrix properties of adjacent components. When the system eigenvectors and eigenvalues are calculated, their accuracy can be assessed by comparing the smallest and largest retained substructure eigenvalues to the smallest and largest eigenvalue from the system modes of interest. In many cases, different substructures are developed, reduced, and integrated by different organizations. Therefore, the ability to reduce each component equation size independently from all other components and to assess the accuracy of the system eigenvectors to be calculated independently of the other components is an important requirement. The method presented here satisfies this twofold requirement; moreover, it can use virtually any set of boundary conditions for the calculation of component modes.

For a linear structural dynamic problem with a proportional damping matrix, the eigenvectors are calculated to decouple the equations of motion. By integrating these decoupled (modal) equations individually and then transforming them back into the original physical coordinate system, the equations of motion are solved. When the vibration of a system structure is described by its homogeneous solution, its substructures are undergoing forced vibration, subjected to boundary forces from adjacent substructures. For this dynamic problem, substructure modal coordinates are vibrating with the same frequency as that of the system mode of excitation (homogeneous solution). As shown in Ref. 1, these substructure modal coordinates are approximated by comparing the associated eigenvalues to the system eigenvalue of excitation; the number of independent substructure modal coordinates is made much smaller than the number of original physical coordinates. Nevertheless, the substructure portion of the homogeneous system solution can be adequately represented by the linear combination of all substructure modes and approximated modal coordinates. Therefore, when substructure stiffness and mass are represented with these approximated substructure modal coordinates, the new representation can retain the properties in the original matrices that are used to describe the substructure portion of the homogeneous solution. Since the homogeneous solution is actually a system eigenvector, the new representation of substructure mass and stiffness with fewer independent variables can retain the properties associated with this system mode.

When the damping matrix is not proportional to the mass or stiffness matrix, second-order equations of motion cannot be decoupled by transformation into this modal coordinate system. Therefore, the equations of motion are converted to the first-order (state vector form) equations of motion introducing independent variables for the velocities.[5] From these converted state vector form equations, complex eigenvectors are calculated and the equations are decoupled. Similar to the problem of proportional damping, substructure modal coordinates are solved from the state vector form equations for homogeneous system solution. These solved substructure modal coordinates are approximated by comparing the magnitudes of substructure and system eigenvalues. The mass, damping, and stiffness of the substructure are represented in the approximated substructure modal coordinates; thus, the new representation of the substructure state vector form equations retains the dynamic properties necessary to describe the substructure portion of the system homogeneous solution (system eigenvector).

This paper discusses the use of three different types of component modes (free-free, hybrid, and cantilever), as well as the first- and second-order approximations of the component modal coordinates. Because of space limitations, detailed discussions are presented only for the first-order approximation and the use of low-frequency component free-free and cantilever modes.

State Vector Form Equation of Motion

First, the state vector form of the equations of motion for a damped structure will be developed. The second-order equations of motion are derived as the Euler equations of a functional. For this functional, displacement time derivatives and velocities are exactly compatible. To develop the state vector form of the equations of motion, new independent variables are introduced for velocities and compatibility is relaxed by means of Lagrangian multipliers. From the Euler equations of the resulting functional, these Lagrangian multipliers are solved and reintroduced into the resulting functional. This modified functional is used to derive first-order (state vector form) finite-element equations of motion in terms of the independent variables (i.e., displacements and velocities). This equation is explored later for the development of component mode synthesis methods.

The equilibrium equations of a structure with velocity proportional forces are given by

$$E_{ijk\ell}u_{j,k\ell} + d_{ij}\dot{u}_j + m\ddot{u}_i - f_i = 0 \tag{1}$$

where u_i, $\dot{u}_i$, and $\ddot{u}_i$ are components of the displacement, velocity, and acceleration vectors respectively and f_i represents the components of the applied force vector. In Eq. (1), the velocity proportional nonconservative forces are given as the product of the damping coefficients d_{ij} and velocities $\dot{u}_j$. This equation is obtained as one of Euler's equations of the following functional:

$$H = \int_{t_1}^{t_2}\int_{\nabla}\left(E_{ijk\ell}u_{i,j}v_{k,l} + d_{\ell m}\dot{u}_\ell v_m + m\dot{u}_\ell\dot{v}_\ell + v_\ell f_\ell + u_\ell g_\ell\right) \mathrm{d}t\,\mathrm{d}v \tag{2}$$

where v_i and g_i lead to an adjoint equilibrium equation of Eq. (1). In the functional H, new independent variables p_i and q_i are introduced for the time derivatives $\dot{u}_i$ and $\dot{v}_i$, respectively, by means of Lagrangian multipliers:

$$p_i + y_i = \dot{u}_i \tag{3}$$

$$q_i + z_i = \dot{v}_i \tag{4}$$

In the preceding equations, differences y_i between the independent variables p_i and $\dot{u}_i$, and differences z_i between q_i and $\dot{v}_i$ are introduced to show that p_i and q_i need not be equal to $\dot{u}_i$ and $\dot{v}_i$, respectively. These differences are prescribed quantities. From Euler's equations of the resulting functional, the Lagrangian multipliers are solved in terms of u_i and p_i, and v_i and q_i. By introducing the solved Lagrangian multipliers into the resulting functional, the following modified functional is obtained:

$$Hm = \int_{t_1}^{t_2}\int_{\nabla}\left(E_{ijk\ell}u_{i,j}v_{k,l} - d_{lm}v_l\dot{u}_m + mq_l\dot{u}_l + mp_\ell\dot{v}_\ell - mp_\ell q_\ell + v_\ell f_\ell + u_\ell g_\ell - mp_\ell z_\ell - mq_\ell y_\ell\right) \mathrm{d}t\,\mathrm{d}v \tag{5}$$

In the case of finite-element formulation, field variables and applied forces are interpolated from the nodal values of these quantities and spatial integrations are performed. From this integrated functional, variations of all independent variables are taken and the coefficients of the variations are equated to zero, resulting in the finite-element state variable form equations of motion. Some of the finite-element equations are compatibility and equilibrium equations involving the variables u_i and p_i. The matrix form of these equations is presented below:

$$[A]\dot{X} + [B]X = F \tag{6}$$

where

$$[A]=\begin{bmatrix}[0] & [M]\\ [M] & [D]\end{bmatrix} \tag{7}$$

$$[B]=\begin{bmatrix}-[M] & [0]\\ [0] & [K]\end{bmatrix} \tag{8}$$

$$X=\begin{Bmatrix}p\\ u\end{Bmatrix} \tag{9}$$

$$F=\begin{Bmatrix}y\\ f\end{Bmatrix} \tag{10}$$

In the above expressions, the mass matrix $[M]$ and stiffness matrix $[K]$ are obtained from spatial integrations of the corresponding terms in Eq. (5). The components of vectors $\boldsymbol{p}$, $\boldsymbol{u}$, $\boldsymbol{y}$, and $\boldsymbol{f}$ can be considered nodal values of p_i, u_i, y_i, and f_i respectively. From the integration of terms involving $d_{\ell m}$ in Eq. (5), the damping matrix $[D]$ is obtained; it is assumed to be symmetric and nonproportional to either the stiffness $[K]$ or the mass matrix $[M]$. In Eq. (10), the vector $\boldsymbol{y}$ is a nonzero prescribed quantity arising from prescribed differences y_i between p_i and $\dot{u}_i$. These nonzero differences are introduced to show that the vector $\boldsymbol{F}$ can be any prescribed vector.

Component Mode Synthesis with Free-Free Modes

The homogeneous solution of a system described by Eq. (6) is given by

$$V=Q\exp(Lt) \tag{11}$$

where $\boldsymbol{Q}$ and L are an eigenvector and eigenvalue of the system. For a component of the system described by Eq. (11), the state vector equation of motion becomes

$$(L[A]+[B])V=F_b \tag{12}$$

where $[A]$ and $[B]$ are component matrices corresponding to Eqs. (7) and (8) respectively. In the preceding equation, subscript b indicates that components of the vector $\boldsymbol{F}$ are nonzero only at interface degrees of freedom. Eigenvector $\boldsymbol{h}$ and eigenvalue ℓ associated with component matrices $[A]$ and $[B]$ satisfy the following relations:

$$[h]^T[A][h]=[I] \tag{13}$$

$$[h]^T[B][h]=[\ell] \tag{14}$$

Equation (12) can be solved as the linear combination of all component eigenvectors, i.e.,

$$V=[h]Z \tag{15}$$

By introducing Eq. (15) into Eq. (12) and premultiplying the resulting equation by a transposed component eigenvector $\boldsymbol{h}^T$, the component Z_i of the independent variable vector $\boldsymbol{Z}$ associated with a pair of eigenvalue ℓ and eigenvector $\boldsymbol{h}$ is given by

$$Z_i=(L+\ell)^{-1}h_b^TF_b \tag{16}$$

This expression is used to approximate an independent variable Z_i to reduce the total number of independent variables.

For this purpose, Eq. (16) is expanded by the Taylor series, as follows:

When $|\ell|\gg|L|$

$${}^HZ_i=\frac{1}{{}^H\ell}\left(1-\frac{L}{{}^H\ell}+O^2\left(\frac{L}{{}^H\ell}\right)\right){}^Hh_b^TF_b \tag{17}$$

When $|\ell|\ll|L|$

$${}^LZ_i=\frac{1}{L}\left(1-\frac{{}^L\ell}{L}+O^2\left(\frac{{}^L\ell}{L}\right)\right){}^Lh_b^TF_b \tag{18}$$

When substructure eigenvalue ℓ is comparable to system eigenvalue L, no approximation is made. In Eqs. (17) and (18), if higher order terms are discarded, Z_i is expressed by the vector $\boldsymbol{F}_b$. Generally, the number of interface degrees of freedom is much smaller than that of the independent variables Z_i associated with smaller or larger component eigenvalues. Thus, the independent variables Z_i can be approximated by fewer degrees of freedom at the interface. These approximated independent variables are to approximate the component portion of the system homogeneous solution through the linear combination of all component eigenvectors.

Taking the first term of Eq. (17) for component independent variables associated with eigenvalues that are significantly larger than the system eigenvalues of interest results in an approximated component solution:

$$V={}^R[h]{}^RZ+{}^C[h]{}^CZ+{}^H[h]{}^H[\ell]^{-1H}[h]_b^TF_b \tag{19}$$

By transforming matrices $[A]$ and $[B]$ by this expression, the size (degrees of freedom) of the component equation is reduced.

Since Eq. (19) involves all component eigenvectors and contains complex numbers, it is not computationally convenient. Therefore, a new expression is derived that does not require the calculation of eigenvectors with larger eigenvalues and contains only real numbers. With the introduction of new independent variables ${}^C\boldsymbol{X}$, the variables ${}^C\boldsymbol{Z}$ may be expressed in a different form:

$${}^CZ={}^CX+{}^C[\ell]^{-1C}[h]^TF_b \tag{20}$$

Equation (20) is introduced into Eq. (19) and rearranged as follows:

$$V={}^R[h]{}^RZ+{}^C[h]{}^CX+{}^e[h]{}^e[\ell]^{-1e}[h]_b^TF_b \tag{21}$$

where matrices ${}^e[h]$ and ${}^e[\ell]$ indicate the association with component nonzero eigenvalues and eigenvectors. The third term of Eq. (21) contains all component eigenvectors and eigenvalues. To express the transformation without using larger eigenvalue eigenvectors, two relations are derived that relate the terms ${}^e[h]{}^e[\ell]^{-1e}[h]$ to $[A]$ and $[B]$.

The component equations are expressed as follows:

$$[A]\dot{V}+[B]V=F \tag{22}$$

Introducing the expression in Eq. (15) into the preceding equations results in the following relation:

$$[A][h]\dot{Z}+[B][h]Z=F \tag{23}$$

Premultiplying Eq. (23) by a transposed eigenvector matrix yields, for modal coordinates associated with nonzero eigenvalues,

$${}^e\dot{Z}+{}^e[\ell]{}^eZ={}^e[h]^TF \tag{24}$$

and for rigid body modes yields

$${}^RZ={}^R[h]^TF \tag{25}$$

The process of solving time derivatives of the modal coordinates from Eqs. (24) and (25), introducing them into Eq. (23),

and rearranging the resulting equation yields

$$\left([A]^R[h]^R[h]^T+[A]^e[h]^e[h]^T\right)F=F \tag{26}$$

In deriving the preceding equation, the following relation is used:

$$[A]^e h^e \ell = -[B]^e h \tag{27}$$

Since vector F in Eq. (26) is arbitrary, the following relation holds:

$$[A][h][h]^T=[I] \tag{28}$$

When ${}^eZ$ of Eq. (24) is solved and introduced along with Eq. (25) into Eq. (23), the results can be arranged as follows:

$$[B]^e[h]^e[\ell]^{-1e}[h]^T+[A]^R[h]^R[h]^T=[I] \tag{29}$$

By introducing Eq. (29) into Eq. (21), a new expression is obtained in terms of the component retained (not approximated) eigenvectors, partitions of $[A]$ and $[B]$ matrices, and interface partitioned solution vector V_b:

$$\begin{Bmatrix} V_i \\ V_b \end{Bmatrix} = \begin{pmatrix} -[B]_{ii}^{-1}\left([A]_{ii}+[A]_{ii}[B]_{ii}^{-1}[B]_{ib}\right)^R[h]_b, & \\ [0], & \end{pmatrix}$$

$$\begin{pmatrix} {}^C[h]+[B]_{ii}^{-1}[B]_{ib}{}^C[h]_b, & -[B]_{ii}^{-1}[B]_{ib} \\ [0], & [I] \end{pmatrix} \begin{Bmatrix} {}^RZ \\ {}^CX \\ V_b \end{Bmatrix} \tag{30}$$

Note that Eq. (30) does not require the calculation of the approximated eigenvectors and associated eigenvalues. This expression can be used for a reduction transformation of substructure matrices $[A]$ and $[B]$. The transformed substructure matrices are coupled with other substructure matrices to construct a system equation. When the system equation is solved with zero y vector, a solution is obtained for a linear transient problem.

Component Mode Synthesis with Cantilever Modes

In this section, the eigenvectors of the internally partitioned $[A]$ and $[B]$ matrices are used to derive reduction transformation equations. Equation (12) is rewritten in the partitioned form as follows:

$$\begin{pmatrix} L[A]_{ii}+[B]_{ii}, & L[A]_{ib}+[B]_{ib} \\ L[A]_{bi}+[B]_{bi}, & L[A]_{bb}+[B]_{bb} \end{pmatrix} \begin{Bmatrix} V_i \\ V_b \end{Bmatrix} = \begin{Bmatrix} O \\ O \end{Bmatrix} \tag{31}$$

Similar to the cantilever modes in the Craig-Bampton method[7] which are natural modes of internally partitioned mass and stiffness matrices, the complex cantilever eigenvector (mode) $\bar{h}$ and eigenvalue $\bar{\ell}$ are calculated from the internally partitioned matrices $[A]_{ii}$ and $[B]_{ii}$. The internally partitioned solution vector V_i is written as a linear combination of these complex cantilever eigenvectors:

$$V_i=[\bar{h}]\bar{Z} \tag{32}$$

where $\bar{Z}$ is a modal coordinate vector associated with eigenvector $[\bar{h}]$.

By introducing Eq. (32) into the upper half of Eq. (31) and premultiplying the resulting equation by the transpose of vector $\bar{h}$, modal coordinates Z may be expressed as follows:

$$\bar{Z}=-(\bar{\ell}+L)^{-1}\bar{h}^T(L[A]_{ib}+[B]_{ib})V_b \tag{33}$$

With this equation, modal coordinates $\bar{Z}_j$ associated with eigenvalues either larger or smaller than the system eigenvalues of interest are approximated by Taylor's expansion. When $|\bar{\ell}| \gg |L|$

$${}^H\bar{Z}=-\frac{1}{H\bar{\ell}}\left(1-\frac{L}{H\bar{\ell}}\right)^{-1}{}^H\bar{h}^T(L[A]_{ib}+[B]_{ib})V_b \tag{34}$$

and when $|\bar{\ell}| \ll |L|$

$$Z=-\frac{1}{L}\left(1+\frac{{}^L\bar{\ell}}{L}\right){}^L\bar{h}^T(L[A]_{ib}+[B]_{ib})V_b \tag{35}$$

When the component eigenvalues $\bar{\ell}$ are comparable to the system eigenvalue L, no approximation is made. The first term of Eq. (34) for the component modal coordinates associated with eigenvalues that are significantly larger than the system eigenvalue of interest is used to express the approximated component solution

$$V_i=[\bar{h}]{}^C\bar{Z}-\left({}^H[\bar{h}]^H[\bar{\ell}]^{-1H}[\bar{h}]^T\right) \times(L[A]_{ib}+[B]_{ib})V_b \tag{36}$$

This expression of reduction transformation equations contains all component eigenvectors. To simplify the form of Eq. (36), the modal coordinates ${}^C\bar{Z}$ may be expressed differently with the introduction of new independent variables ${}^C\bar{X}$:

$${}^C\bar{Z}={}^C\bar{X}-{}^C[\bar{\ell}]^{-1C}[\bar{h}]^T(L[A]_{ib}+[B]_{ib})V_b \tag{37}$$

The result of introducing Eq. (37) into Eq. (36) and rearranging is

$$V_i={}^C[\bar{h}]{}^C\bar{X}-[\bar{h}][\bar{\ell}]^{-1}[\bar{h}]^T(L[A]_{ib}+[B]_{ib})V_b \tag{38}$$

Matrix $[B]_{ii}$, eigenvalues $[\bar{\ell}]$, and eigenvectors $[\bar{h}]$ satisfy the following relation:

$$[B]_{ii}[\bar{h}][\bar{\ell}]^{-1}[\bar{h}]^T=[I] \tag{39}$$

By introducing Eq. (39) into Eq. (38), a new expression is obtained in terms of the component retained cantilever eigenvectors (low-frequency modes), partitions of matrices $[A]$ and $[B]$, and boundary partitioned solution vector V:

$$\begin{Bmatrix} V_i \\ V_b \end{Bmatrix} = \begin{pmatrix} {}^C[\bar{h}], & -[B]_{ii}^{-1}[A]_{ib}, & -[B]_{ii}^{-1}[B]_{ib} \\ [0], & [0], & [I] \end{pmatrix} \begin{Bmatrix} {}^C\bar{X} \\ \dot{V}_b \\ V_b \end{Bmatrix} \tag{40}$$

where $\dot{V}_b$ is a vector of new independent variables defined by

$$\dot{V}_b=LV_b \tag{41}$$

Similar to the manipulation demonstrated in Ref. 1, the expressions for reduction transformation equations can be obtained by retaining the component cantilever modes associated with any range of component eigenvalues or by approximating modal coordinates up to the second order.

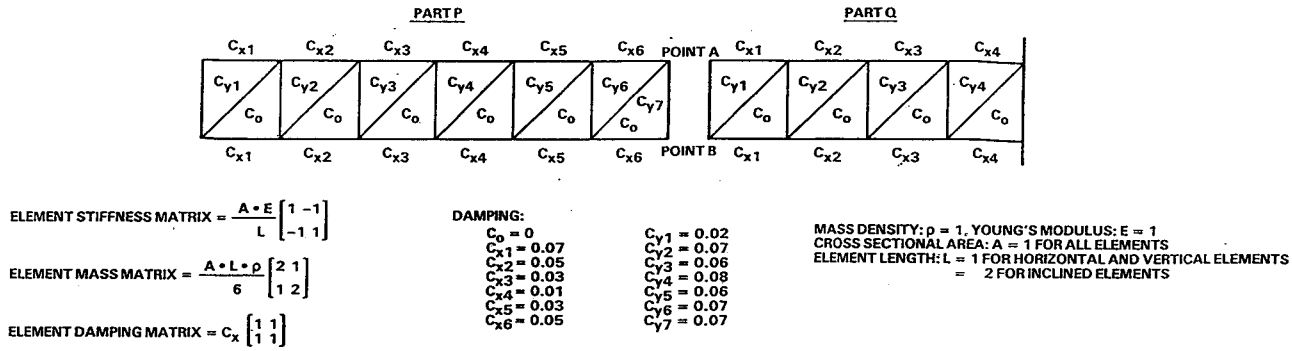

Fig. 1 Example structure.

Table 1 Complex part of system eigenvalues

No.	Exact 80 DOF	Free-free $Wd = 1.523$ 28 modes, 32 I/F DOF	Cantilever $Wd = 1.461$ 24 modes, 32 I/F DOF (6 v)	Cantilever $Wd = 1.461$ 24 modes, 32 I/F DOF (modified)
1	0.002713	0.002713	0.002713	0.002713
2	0.127053	0.127053	0.127056	0.127197
4	0.064175	0.064175	0.064176	0.064189
6	0.11733	0.11733	0.11733	0.11790
8	0.18476	0.18476	0.18476	0.18492
10	0.31015	0.31016	0.31015	0.31063
12	0.41798	0.41799	0.41811	0.42318
14	0.42739	0.42742	0.42743	0.42318
16	0.57431	0.57431	0.57431	0.57451
18	0.65718	0.65723	0.65719	0.75894
20	0.70754	0.70783	0.70754	0.70760
22	0.73823	0.73825	0.73850	0.74384
24	0.81171	0.81172	0.81174	0.81296
26	0.86905	0.86912	0.86915	0.87204
28	0.90427	0.90435	0.90449	0.90728
30	0.95233	0.95225	0.95225	0.95254
32	0.98513	0.98647	0.98549	0.99207
34	1.12731	1.12804	1.12829	1.14460
36	1.25595	1.25626	1.25718	1.27038
38	1.42436	1.42841	1.42880	1.44947
40	1.51974	1.56917	1.52325	1.53495
42	1.69674	1.71793	1.71616	1.74338
44	1.70699	1.78290	1.78665	1.88831
46	1.72863	1.86622	1.84377	2.17217
48	1.79717	2.04390	2.04700	2.73706
50	1.86680	2.49020	2.47486	2.90860
52	2.09889	2.77589	2.77427	2.94264
54	2.37794	2.94374	2.88925	3.07394
56	2.69152	3.00144	2.94784	3.21911
58	2.77244	3.08462	3.04384	
60	2.84528	3.25122	3.10722	
62	2.85926		3.30015	
⋮				
80	3.44441			

Discussion

Reduction transformation equations have been presented for use in low-frequency component complex free-free and cantilever modes. When the reduction transformation equations are directly applied to the component first-order equations of motion, the resulting matrices (reduced $[A]$ and $[B]$ matrices) in the equations become complex numbers. To obtain real-number matrices, the complex number modal coordinates are transformed to their real and imaginary components.

Although discussions have been presented only for the use of free-free and cantilever modes, these processes can easily be extended to derive reduction transformation equations for complex hybrid modes. A hybrid mode is a natural mode of the component in which some interface coordinates are free and others are fixed. For hybrid modes, a procedure similar to the one employed for complex cantilever modes is repeated with the following two assumptions: 1) subscript i of Eq. (31) is understood to be the sum of the internal and free interface degrees of freedom, and 2) the relation in Eq. (29) is used instead of that in Eq. (39), with the understanding that the $[A]$ and $[B]$ matrices of Eq. (29) are partitioned to the DOF sum.

In a method similar to that used for a structure with a proportional damping matrix,[1] reduction transformation equations can be derived to retain any eigenvalue range. This is done by approximating the modal coordinates associated with the larger or smaller component eigenvalues, as compared to system eigenvalues through Eqs. (17) and (18) or (34) and (35). These approximated modal coordinates are then introduced into the expression for the component state variables (the linear combination of all component modes with linear coefficients understood to be modal coordinates). Also, the use of higher-order terms in the Taylor series can be incorporated into the development of component reduction transformation equations by introducing terms, including higher-order terms in Eqs. (17), (18), (34), and (35), into the expression of component state variables.

As shown in the examples for proportional damping matrix structures,[1] the accuracy of the calculated system eigenvalues depends on the relative magnitudes of the retained component eigenvalues and calculated system eigenvalues. In the case of calculating low-frequency system modes, the larger the retained component eigenvalues, the more accurate the low-frequency system mode becomes. Since component complex modal coordinates are approximated by comparing these eigenvalues, the same accuracy dependence on relative magnitudes is expected for the damped structures. Because the coordinate system spanned by transformation Eq. (30) and the reduced matrices of Chung and Craig[3] are the same representation of a component, the same degree of improvement achieved by Chung and Craig can be expected in the matrices reduced by Eq. (30). Furthermore, the same degree of approximation used to derive Eq. (30) is embedded in the reduction transformation Eq. (40) that uses complex cantilever modes. Therefore, the same degree of accuracy should be obtained by both methods with free-free modes [Eq. (30)] and cantilever modes [Eq. (40)] to calculate system eigenvalues and eigenvectors. These accuracy features can be recognized in the results of example problems in this paper.

The disadvantages of using component eigenvectors have been documented:[5] 1) the associated computational burden is enormous, 2) the eigenvectors are generally complex quantities that lead to complex assembled system matrices, and 3) the boundary conditions to be imposed in eigenproblems are highly ambiguous, i.e., whether the internal boundary points should be fixed, free, or loaded in some fashion. The first disadvantage may not be so for solutions of only the low-frequency component modes from component eigenproblems smaller than the system eigenproblems. From the discussion

previously presented in this paper, the second disadvantage apparently does not hold, and, because any type of boundary condition can be employed to calculate substructure modes, the third disadvantage is not true.

The so-called subspace iteration technique for an eigenproblem with several trial vectors is a method that iteratively seeks a subspace in which the eigenvectors of interest lie within a prescribed error. Different from the iterative techniques, the component mode synthesis is the method for obtaining the subspace in which the substructure portions of system modes of interest lie with the accuracy assessment, without any iteration or information from adjacent substructures. When component modal coordinates are approximated by taking the first term of Eq. (17), errors in the approximation are orders of $|L|/|\ell|$, where $|L|$ is the eigenvalue of the system eigenvector of interest and $|\ell|$ is the eigenvalue of the component eigenvector with the highest frequency. With these approximated component modal coordinates, a new reduced subspace is spanned in which the component portion of the system eigenvector lies within errors of the order $|L|/|\ell|$. In this way, the accuracy of the system eigenvector to be calculated can be assessed without any information from adjacent components.

Example

The rod structure in Fig. 1 is an example problem to demonstrate numerically the validity of the methods presented in this paper. The structure consists of two components, P and Q. Each element has extensional stiffness, distributed mass, and damping, shown in the illustration in terms of element stiffness, damping, and mass matrices. Each nodal point has two translational freedoms, x and y. Part P consists of 28 nodal points and 56 state variables (28 displacements and 28 velocities) with 8 interface degrees of freedom. Part Q consists of 16 nodal points and 32 state variables, also with 8 interface degrees of freedom. To assess the accuracy of different component mode representations, only part P is represented with substructure modes and then is coupled to part Q at nodal points A and B. The complex part (Wd) of the eigenvalue $(S + iWd)$ is used as a measure of accuracy.

Table 1 shows the complex part of the system eigenvalues except for the first two eigenvalues, which are real numbers. Since eigenvalues appear as complex conjugate pairs, only one of them is tabulated. For reference, the eigenvalues of the system structure constructed with all physical degrees of freedom are shown in the first column of Table 1. The second column lists the eigenvalues of the system structure constructed by reducing part P with Eq. (29). In the reduction transformation equation, 28 free-free modes of the complex part of the eigenvalues up to about 1.5 are used, resulting in system matrices with 60 degrees of freedom.

The third column of Table 1 shows the eigenvalues of the 62 DOF system structure, with part P reduced by Eq. (40). In the reduction transformation matrix, 24 cantilever modes, also of a value up to about 1.5, of the complex part of the eigenvalues are used. Discarding $\dot{V}_b$ from Eq. (40) reduces the matrices for part P and couples them to part Q.

In the fourth column, eigenvalues of the constructed 56 DOF system structure are shown. Note that the calculated system eigenvalues, which are close to or smaller than the retained substructure eigenvalues, agree well with those of the first column. Moreover, the eigenvalues calculated by modifying Eq. (40) (fourth column) are less accurate than those in the third column, which use Eq. (40) without modification.

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